Activity 2: Chaos in a simple system: Part I

In this activity, we study a very simple model of population dynamics. This model is known as the *logistic map*. Although the model is exceptionally simple, it's behavior can be unpredictable. And yet, despite the unpredictability of this and other chaotic systems, we'll see in (in the next activity) that there is some kind of order hidden in chaotic dynamics.

To make our activity concrete, we'll use a contrived example of mosquitos on an isolated island. We'll explore a simple mathematical model that mimics the seasonal variation of the mosquito population.

1. Imagine that, in the first season, there are n_1 mosquitos on the island; n is the number of mosquitos and 1 represents the first season. Similarly the number of mosquitos in the second season would be n_2, and so forth. The number of mosquitos in one season affects the number of mosquitos in the next season (more mosquitos lay more eggs, leading to an increased number in the following season). In math, we can represent such a relationship in the following way:

$$n_{i+1} = rn_i$$

   In this equation, r represents the rate at which the populations changes. If $r > 1$ what will happen to the number of mosquitos after many seasons? What will happen if $r < 1$? Is there a value of r that makes sense? That is, is there a value of r that makes the model a good one? Answer these questions before going on.

2. The model isn't a very good one because for any value of $r > 1$, the population will grow without bound (which is obviously unrealistic). For any value $r < 1$, the mosquitos will always die out.

   A more realistic model would recognize that there is some maximum amount of mosquitos that the island support (call it N) and when the actual number of mosquitoes is small, there are lots of resources (food to eat) so the population will grow rapidly. On the other hand, when the number of mosquitos is close to the maximum (in math, $n_i/N \sim 1$) many will die without reproducing because they will starve.

   If we let $x_i = n_i/N$ represent the current population as a fraction of the maximum possible, then we could express this more complex model as:

$$x_{i+1} = rx_i(1 - x_i)$$

   The way this equation works x_i will always be between 0 and 1 (as it should be), as long as the starting value x_0 is between 0 and 1. We also assume that $0 \leq r < 4$. Explain in words how this model has the features we just described.

3. By hand or with your calculator, calculate the first ten values of x (i.e., up to x_{10}) assuming that $r < 1$ (pick any value). What happens? Does the answer depend on your starting value of the population, x_0?

4. Do the process again but with $1 < r < 2$. What is the ultimate result? Does it depend on x_0?

5. Use the MATLAB program `logistic_map.m` to repeat the previous two exercises. You can play around with the values of r, the number of steps, etc., in the MATLAB program.

6. What happens for values of r bigger than 3 but smaller than 3.4? Does this make sense? What does this mean for the mosquito population? Explain.