

Solutions

(15 pts) **Problem 1:** Multiple choice conceptual questions. Choose the best answer. **Fill in your answers on the bubble sheet.**

1.1. You have two balloons, one filled with air and one filled with helium. If you put both balloons into a tub of liquid nitrogen, which one will end up with the *smallest* volume?

- a. the air balloon *because most of air will condense into liquid (we did demo in class)*
- b. the helium balloon
- c. they will end up with the same volume

1.2. The second law of thermodynamics is a statement of:

- a. conservation of energy
- b. conservation of (regular) momentum
- c. conservation of angular momentum
- d. conservation of mass/volume
- e. probability *It's not impossible for heat to flow from cold to hot, it's just very VERY unlikely. (Deck of cards example from class)*

1.3. As I'm taking data in my lab, I often take data from a light detector and average the data for 1 second for each data point. You can assume that this represents data from 1,000,000 photons, for example. Suppose I decide to average the data for 2 seconds per point instead (collecting 2,000,000 photons, for example). How much better is my signal-to-noise ratio likely to be? Hint: Think about the statistical fluctuations (the "noise") likely to occur in each case.

- a. the same
 - b. $\sqrt{2}$ times better
 - c. 2 times better
 - d. 4 times better
 - e. 8 times better
- fluctuations = 1000*
 $S/N = \frac{1000000}{1000} = 1000$
- fluctuations = 1000 $\sqrt{2}$*
 $S/N = \frac{2000000}{1000\sqrt{2}} = 1000\sqrt{2}$
- ratio of S/N: $\frac{1000\sqrt{2}}{1000} = \sqrt{2}$*

1.4. In the "ladies belt demo" (the belt was like a "closed-closed" string), suppose the fundamental frequency is seen at 500 Hz. What frequency will have five antinodes?

- a. 800 Hz
 - b. 1250
 - c. 1500
 - d. 2000
 - e. 2500
 - f. 3000 Hz
- fifth harmonic, $f_5 = 5f_1$*

1.5. When a "wave" is actually a finite duration pulse, ^{*in a medium w/ dispersion*} the group velocity is the speed at which:

- a. the overall pulse envelope propagates
- b. all individual frequency components propagate *→ there is no such velocity in a medium w/ dispersion*
- c. the average frequency component propagates *→ this would be the phase velocity*
- d. none of the above
- e. more than one of the above

1.6. You should have found the wave speed of waves on your slinky to be directly proportional to the slinky's length. (This was true for both longitudinal and transverse waves.) Compare the time it takes waves to do a round-trip path when the slinky is stretched to 5 feet compared to when it is stretched to 10 feet.

- a. $t_{5ft} > t_{10ft}$
 - b. $t_{5ft} \approx t_{10ft}$
 - c. $t_{5ft} < t_{10ft}$
- $t = \frac{x}{v}$, so if x and v are both doubled t stays the same.*
(Hopefully you observed this.)

1.7. Complete the following based on our in-class discussions: To localize a wave in space, you need lots of _____

- a. amplitude
- b. energy
- c. patience
- d. phase
- e. wavenumbers *... and to localize it in time, you need lots of frequencies.*

1.8. The sound of a trumpet playing a note is qualitatively different than the sound of a flute playing the same note. Why is that?

- a. The two notes have different amplitudes.
 - b. The two notes have different durations.
 - c. The two notes have different fundamental frequencies.
 - d. The two notes have different phases.
 - e. The two notes have different strengths of harmonics.
- Different shape of wave → different Fourier coefficients*

$$A \cos(kx - \omega t + \phi)$$

1.9. The wavefunction for a particular wave on a string is given by: $f(x,t) = 4 \cos(2x - 6t + 5)$. You may assume that the numbers in that equation and in the answer choices below are all given in terms of the appropriate SI units. What is the amplitude?

- | | | |
|---------------------------------------|-------------|-------------|
| a. 2 | e. 6 | i. $2\pi/5$ |
| b. 3 | f. $2\pi/2$ | j. $2\pi/6$ |
| <input checked="" type="radio"/> c. 4 | g. $2\pi/3$ | |
| d. 5 | h. $2\pi/4$ | |

A

1.10. Same equation. What is the wavelength?

- | | | |
|------|--|-------------|
| a. 2 | e. 6 | i. $2\pi/5$ |
| b. 3 | <input checked="" type="radio"/> f. $2\pi/2$ | j. $2\pi/6$ |
| c. 4 | g. $2\pi/3$ | |
| d. 5 | h. $2\pi/4$ | |

$\lambda = \frac{2\pi}{k}$
 $k = 2$

1.11. Same equation. What is the wavenumber (k)?

- | | | |
|---------------------------------------|-------------|-------------|
| <input checked="" type="radio"/> a. 2 | e. 6 | i. $2\pi/5$ |
| b. 3 | f. $2\pi/2$ | j. $2\pi/6$ |
| c. 4 | g. $2\pi/3$ | |
| d. 5 | h. $2\pi/4$ | |

1.12. Same equation. What is the period?

- | | | |
|------|-------------|--|
| a. 2 | e. 6 | i. $2\pi/5$ |
| b. 3 | f. $2\pi/2$ | <input checked="" type="radio"/> j. $2\pi/6$ |
| c. 4 | g. $2\pi/3$ | |
| d. 5 | h. $2\pi/4$ | |

$T = 2\pi/\omega$
 $\omega = 6$

1.13. Same equation. What is the angular frequency (ω)?

- | | | |
|------|---------------------------------------|-------------|
| a. 2 | <input checked="" type="radio"/> e. 6 | i. $2\pi/5$ |
| b. 3 | f. $2\pi/2$ | j. $2\pi/6$ |
| c. 4 | g. $2\pi/3$ | |
| d. 5 | h. $2\pi/4$ | |

1.14. Same equation. What is the phase (ϕ)?

- | | | |
|---------------------------------------|-------------|-------------|
| a. 2 | e. 6 | i. $2\pi/5$ |
| b. 3 | f. $2\pi/2$ | j. $2\pi/6$ |
| c. 4 | g. $2\pi/3$ | |
| <input checked="" type="radio"/> d. 5 | h. $2\pi/4$ | |

1.15. Same equation. What is the wave's speed?

- | | | |
|---------------------------------------|-------------|-------------|
| a. 2 | e. 6 | i. $2\pi/5$ |
| <input checked="" type="radio"/> b. 3 | f. $2\pi/2$ | j. $2\pi/6$ |
| c. 4 | g. $2\pi/3$ | |
| d. 5 | h. $2\pi/4$ | |

$v = \frac{\omega}{k} = \frac{6}{2} = 3$

or, just re-write equation as
 $4 \cos[2(x - 3t) + 5]$
 ↑
 this is (obviously?) v .

(11 pts) **Problem 2.** (a) A car (traveling at 44 m/s) is chasing you on your bike (traveling at 12 m/s). The car driver honks her horn and emits a tone of 500 Hz. Use 343 m/s for the speed of sound. What frequency do you hear?

$$f' = f \frac{v \pm v_o}{v \pm v_s}$$

$$= 500 \text{ Hz} \left(\frac{343 - 12}{343 - 44} \right)$$

5 pts

$$= \boxed{553.5 \text{ Hz}}$$

≈ all units of m/s cancel

minus sign in numerator because observer is moving away from source (frequency shifts down)

minus sign in denominator because source is moving towards observer (frequency shifts up)

(b) The animated gif I showed in class where the group and phase velocities were in opposite directions had the following dispersion relation: $\omega = -1/k^2$. (You can assume that the number "1" in the equation has the appropriate units to make ω be in rad/s and k in rad/m.) It was made up of 21 frequency components centered around $k = 1$ rad/m. Find the group and phase velocities of the pulse.

$$\omega = -1/k^2 = -k^{-2}$$

$$v_{\text{group}} = \left. \frac{d\omega}{dk} \right|_{k=\bar{k}} = -(-2k^{-3}) \Big|_{k=1} = 2$$

3 pts

$$v_{\text{group}} = \boxed{2 \text{ m/s}}$$

$$v_{\text{phase}} = \frac{\omega}{k} = \frac{-1/k^2}{k} = -\frac{1}{k^3}$$

3 pts

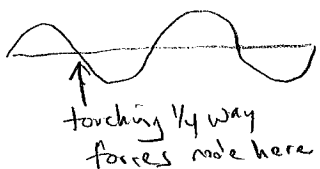
also evaluated at $k = \bar{k}$
 $= -1$

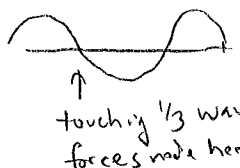
$$v_{\text{phase}} = \boxed{-1 \text{ m/s}}$$

(10 pts) **Problem 3.** Guitar players often tune their instrument using harmonics. Imagine that I use an electronic tuner to tune my low E string to 164.814 Hz, precisely the correct frequency for an equal temperament scale referenced to the standard frequency of 440.000 Hz for the A above middle C. Now I tune my next string, the A string—which is 5 half-steps higher in frequency, a musical "fourth"—using harmonics. I do this by lightly touching the E string 1/4 of the way from the end of the string and lightly touching the A string 1/3 of the way from the end of the string and adjusting its frequency so that the beats go away.

5 pts

(a) What is the frequency ratio of the two notes and why does this tuning method work?

E string:  $f_4 = 4f_1 = 4(164.814 \text{ Hz})$

A string:  $f_3 = 3f_1$

Tuning away beats $\rightarrow (f_4)_{\text{E string}} = (f_3)_{\text{A string}}$
 $4(164.814 \text{ Hz}) = 3(f_1)_{\text{A string}}$

$(f_1)_{\text{A string}} = 219.752 \text{ Hz}$, and freq ratio = $\frac{4}{3}$

5 pts

(b) What is the difference in frequency between an A tuned that way and an A tuned according to the equal temperament scale?

Equal temperament A will be 5 half-steps higher in frequency than that

$$(f_1)_A = (\sqrt[12]{2})^5 \times 164.814 \text{ Hz}$$

$$= 220 \text{ Hz}$$

(or you could just say this will be $\frac{1}{2}$ of freq of A440 Hz since it's down an octave from there.)

$$\Delta f = 220 \text{ Hz} - 219.752 \text{ Hz}$$

$$\approx \boxed{0.248 \text{ Hz}}$$

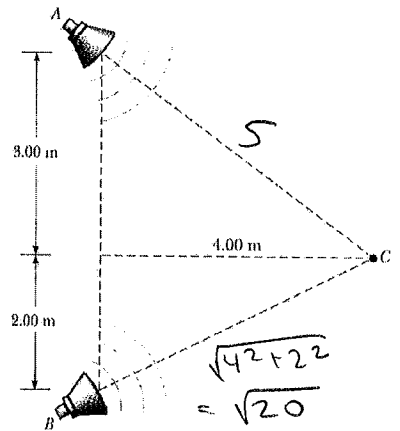
(11 pts) **Problem 4.** Two small speakers emit spherical waves, both with outputs of 2 mW of sound power.

$$I = \frac{P}{4\pi r^2}$$

(a) Assuming the two waves are incoherent so that the *intensities* add, determine the intensity (W/m^2) and sound level (dB) experienced by an observer located at point C.

$$I_C = I_A + I_B = \frac{2 \times 10^{-3} \text{ W}}{4\pi(5 \text{ m})^2} + \frac{2 \times 10^{-3} \text{ W}}{4\pi(\sqrt{20} \text{ m})^2} = 6.366 \cdot 10^{-6} \frac{\text{W}}{\text{m}^2} + 7.9577 \cdot 10^{-6} \frac{\text{W}}{\text{m}^2}$$

$$= \boxed{1.43 \times 10^{-5} \text{ W/m}^2}$$



3 pt

intensity = _____ W/m^2

$$\beta = 10 \log\left(\frac{I}{I_0}\right) = 10 \log\left(\frac{1.43 \times 10^{-5} \frac{\text{W}}{\text{m}^2}}{10^{-12} \frac{\text{W}}{\text{m}^2}}\right) = \boxed{71.56 \text{ dB}}$$

3 pt

sound level = _____ dB

(b) Assuming the two waves are coherent, so that the *amplitudes* add (to produce a pattern of minima and maxima), what is the lowest frequency that will produce a maximum at point C? The two speakers are governed by sinusoidal voltage sources that have the same frequency and are in phase with each other. Use $v_{\text{sound}} = 343 \text{ m/s}$.

$$\text{Maxima: } \Delta \text{Path length} = n\lambda = n\left(\frac{v}{f}\right)$$

$$\text{lowest freq} \rightarrow n=1$$

$$5 \text{ m} - \sqrt{20} \text{ m} = \frac{343 \text{ m/s}}{f}$$

$$f = \frac{343}{5 - \sqrt{20}} \text{ Hz}$$

$$= \boxed{649.8 \text{ Hz}}$$

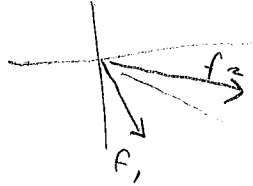
5 pt

(12 pts) **Problem 5.** Add these two cosine functions together using complex number techniques to determine the amplitude and phase of the resulting function: $f_1 = 3\cos(4t + 5)$, $f_2 = 4\cos(4t + 6)$, and thereby write down the resulting function. The numbers "4" and "6" have units of rad/s and radians, respectively. Hint: be careful as to whether your calculator is in degrees or in radians mode. (If you know the method, you don't necessarily have to write down any complex numbers to solve this problem.)

As discussed in class, adding amplitudes + phases of sinusoidal functions is just like adding polar coordinates of vectors.

$$f_1 = 3 \angle 5 \text{ rad} = 3 \angle 286.5^\circ$$

$$f_2 = 4 \angle 6 \text{ rad} = 4 \angle 343.8^\circ$$



$f_1 + f_2$ will be something like this

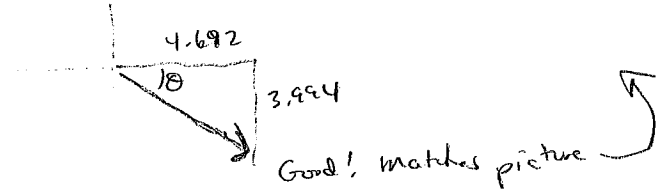
$f_1 + f_2 =$

$$x\text{-coord} = 3\cos 286.5^\circ + 4\cos 343.8^\circ$$

$$= 4.692$$

$$y\text{-coord} = 3\sin 286.5^\circ + 4\sin 343.8^\circ$$

$$= -3.994$$



$$\text{amplitude} = \sqrt{4.692^2 + 3.994^2}$$

$$= \boxed{6.162}$$

$$\text{phase} = -\tan^{-1}\left(\frac{3.994}{4.692}\right)$$

$$= \boxed{-40.41^\circ} = \boxed{-.7053 \text{ rad}}$$

$$\text{resulting function: } \boxed{f(t) = 6.162 \cos(4t - .7053)}$$

Amplitude = _____

Phase = _____ radians, = _____ degrees

Resulting function $f(t) = f_1(t) + f_2(t) =$ _____

(8 pts) **Problem 6.** A sound wave travels from air ($v = 343$ m/s) into a small lake ($v = 1493$ m/s), coming from a source far from the lake, and directly above it, so that the wave crests are essentially parallel to the water as it enters. This is thus a 1D situation, where the same equations we derived for waves on a string are applicable.

(a) What fraction of the incident wave's amplitude is reflected?

$$r = \frac{v_2 - v_1}{v_1 + v_2} = \frac{1493 - 343}{343 + 1493} \frac{\text{m/s}}{\text{m/s}} = \boxed{.626}$$

(b) What fraction of the incident wave's amplitude is transmitted? Hint: don't be surprised if this result... well, surprises you.

$$t = \frac{2v_2}{v_1 + v_2} = \frac{2(1493)}{343 + 1493} = \boxed{1.626}$$

A little surprising, perhaps \rightarrow more amplitude is transmitted than comes in, in the first place. But that's OK, energy is still conserved.

This happens because the wave doesn't flip to

(c) What fraction of the incident wave's power is reflected? *negative when it is reflected.*

$$R = |r|^2 = \boxed{.392}$$

39.2% of energy is reflected

(d) What fraction of the incident wave's power is transmitted?

$$T = 1 - |r|^2 = \boxed{.608}$$

60.8% of energy is transmitted.

(Energy is conserved, as mentioned above.)

(12 pts) **Problem 7.** This equation:

$$\frac{d^2x}{dt^2} = -\frac{\gamma}{m} \frac{dx}{dt} - \frac{k}{m} x$$

has a solution (meaning a function for which the equation is true) of the form $x(t) = Ae^{-t/\tau} \cos(\omega t)$. That solution represents, for example, a spring with an oscillating mass hanging from it, where the amplitude of oscillation continually decreases due to air resistance.

By representing $x(t)$ as a complex exponential, $x(t) = Ae^{t(-\frac{1}{\tau} + i\omega)}$ and plugging it into the equation, find what τ and ω must be in terms of k , m , and γ , to make the equation true. (Yes, this is the same as your homework problem HW 16-5. Demonstrate that you can do the problem, don't just quote the answer to that problem.)

$$x(t) = Ae^{t(-\frac{1}{\tau} + i\omega)}$$

$$\frac{dx}{dt} = \left(-\frac{1}{\tau} + i\omega\right) Ae^{t(-\frac{1}{\tau} + i\omega)}$$

$$\frac{d^2x}{dt^2} = \left(-\frac{1}{\tau} + i\omega\right)^2 Ae^{t(-\frac{1}{\tau} + i\omega)}$$

2 pts
 Plug into equation: $\left(-\frac{1}{\tau} + i\omega\right)^2 Ae^{t(-\frac{1}{\tau} + i\omega)} = -\frac{\gamma}{m} \left[\left(-\frac{1}{\tau} + i\omega\right) Ae^{t(-\frac{1}{\tau} + i\omega)} \right] - \frac{k}{m} Ae^{t(-\frac{1}{\tau} + i\omega)}$

$$\left(-\frac{1}{\tau} + i\omega\right)^2 = -\frac{\gamma}{m} \left(-\frac{1}{\tau} + i\omega\right) - \frac{k}{m}$$

$$\frac{1}{\tau^2} - \frac{2i\omega}{\tau} - \omega^2 = +\frac{\gamma}{m\tau} - \frac{\gamma\omega}{m}i - \frac{k}{m}$$

3 pts
 Imag part of LHS = Imag part of RHS

$$-\frac{2\omega\gamma}{\tau} = -\frac{\gamma\omega}{m} \rightarrow \boxed{\tau = \frac{2m}{\gamma}}$$

Real part of LHS = Real part of RHS

$$\frac{1}{\tau^2} - \omega^2 = \frac{\gamma}{m\tau} - \frac{k}{m}$$

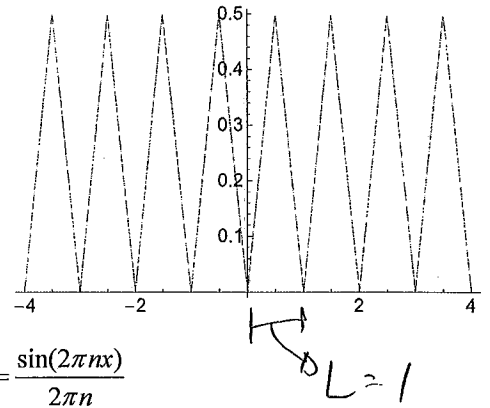
Plug in for τ

$$\frac{\gamma^2}{4m^2} - \omega^2 = \frac{\gamma}{m} \left(\frac{\gamma}{2m}\right) - \frac{k}{m}$$

$$\omega^2 = \frac{\gamma^2}{m^2} \left(\frac{1}{4} - \frac{1}{2}\right) + \frac{k}{m}$$

$$\boxed{\omega = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}}$$

(15 pts) **Problem 8.** In my lab I have a voltage "function generator" which can produce triangular-shaped voltage patterns such as the one shown. The x-axis is seconds; the y-axis is volts. (You can ignore units for the rest of this problem.) From 0 to 0.5 s it sends out a signal that is $f(x) = x$. From 0.5 s to 1.0 s it sends out a signal that is $f(x) = 1 - x$. Then the signal repeats.



Determine the Fourier coefficients of this function and write $f(x)$ as a sum of sines, cosines, and a constant term, as applicable. If any of the Fourier coefficients have obvious values, state your reasons why if they are obvious.

Potentially useful integrals:

$$\int \sin(2\pi nx) dx = -\frac{\cos(2\pi nx)}{2\pi n}$$

$$\int \cos(2\pi nx) dx = \frac{\sin(2\pi nx)}{2\pi n}$$

$$\int x \sin(2\pi nx) dx = -\frac{x \cos(2\pi nx)}{2\pi n} + \frac{\sin(2\pi nx)}{4\pi^2 n^2}$$

$$\int x \cos(2\pi nx) dx = \frac{\cos(2\pi nx)}{4\pi^2 n^2} + \frac{x \sin(2\pi nx)}{2\pi n}$$

Note: although you should recognize that $\sin(n\pi)$ and $\sin(2n\pi) = 0$ for all integer values of n , you can leave your answers in terms of $\cos(n\pi)$ and $\cos(2n\pi)$; that is, you don't have to work out the pattern of your answers in terms of odds/evens or (-1) to some power.

$a_0 = \text{ave value} = 0.25$ (can tell by looking)

$b_n = 0$ since $f(x)$ is even

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{2\pi nx}{L} dx$$

$$= 2 \left[\int_0^{1/2} x \cos 2\pi nx dx + \int_{1/2}^1 (1-x) \cos 2\pi nx dx \right]$$

$$= 2 \left[\left. \frac{\cos 2\pi nx}{4\pi^2 n^2} + x \frac{\sin 2\pi nx}{2\pi n} \right|_0^{1/2} + \frac{\sin 2\pi nx}{2\pi n} \right|_{1/2}^1 - \left(\left. \frac{\cos 2\pi nx}{4\pi^2 n^2} + x \frac{\sin 2\pi nx}{2\pi n} \right|_{1/2}^1 \right) \right]$$

$$= 2 \left[\frac{\cos \pi n - 1}{4\pi^2 n^2} - \left(\frac{\cos 2\pi n - \cos \pi n}{4\pi^2 n^2} \right) \right]$$

$$a_n = \frac{1}{2\pi^2 n^2} [2 \cos \pi n - \cos 2\pi n - 1]$$

→ Simplified enough!

or, to simplify more, $\cos 2\pi n = 1$ always, so brackets = $[2 \cos \pi n - 2]$

$$a_n = \frac{1}{\pi^2 n^2} [\cos \pi n - 1]$$

3 pts a_0 (constant term) = 0.25

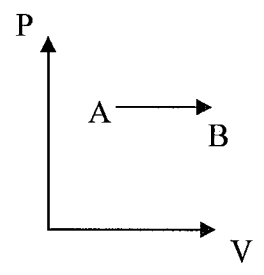
6 pts a_n (cosine coeff) = $\frac{1}{\pi^2 n^2} [\cos \pi n - 1]$

3 pts b_n (sine coeff) = 0

3 pts $f(x) = 0.25 + \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} [\cos \pi n - 1] \cos(2\pi nx)$
(function written as a sum of sines and cosines)

(6 pts, no partial credit) **Problem 9.** Heat is added to 4 moles of a monatomic ideal gas at 300K, while keeping its pressure constant as shown in the diagram. Heat comes in from a reservoir kept at 710 K, which causes the temperature of the gas to increase to 700K as shown in the P-V diagram. What was the change in entropy of the universe during this process? (Hint: find the change in entropy for the gas and for the reservoir separately, then add them together. By the 2nd Law, your final answer must be positive. You can assume that the heat lost by the reservoir is equal to the heat added to the gas.)

$$C_p = \frac{5}{2} R$$



$$\Delta S_{\text{gas}} = \int \frac{dQ}{T} = \int \frac{n C_p dT}{T} = n C_p \ln T_f / T_0$$

$$\Delta S_{\text{reservoir}} = \int \frac{dQ}{T} = \frac{1}{T_{\text{res}}} \int dQ = \frac{Q}{T_{\text{res}}}$$

$$Q = - Q_{\text{added to gas}} = - n C_p \Delta T$$

$$\Delta S_{\text{total}} = \Delta S_{\text{gas}} + \Delta S_{\text{reservoir}}$$

$$= n C_p \ln T_f / T_0 - \frac{n C_p \Delta T}{T_{\text{res}}}$$

$$= 4 \left(\frac{5}{2} \times 8.31 \right) \ln \left(\frac{700}{300} \right) - 4 \left(\frac{5}{2} \times 8.31 \right) \frac{(400)}{710} \quad \text{J/K}$$

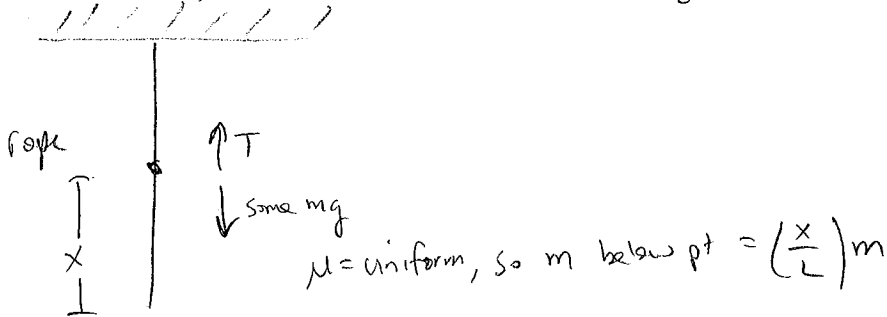
$$= 70.41 - 46.82 \quad \text{J/K}$$

$$= \boxed{23.59 \text{ J/K}}$$

(4 pts, no partial credit) **Extra Credit.**

Suppose you attach a rope (mass m , length L) to the ceiling and let it dangle down so that it just barely reaches the floor without touching. How long will it take for a transverse pulse to travel from the bottom of the rope to the top? Put your answer in terms of m , g , and L .

Hint: You can do this in two steps. First, pick a point on the rope a distance x up from the bottom of the rope, and draw a free-body diagram for that point. There's some weight (but not all the weight) pulling down, and some tension pulling up. That should give you tension as a function of distance. You may assume that the rope's linear mass density will not vary with height. You already know how the wave speed depends on tension and linear mass density, so you should then be able to figure out the wave speed as a function of distance. Then, use that information and some calculus to figure out the answer to the problem.



$$T = \left(\frac{x}{L}m\right)g$$

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{\left(\frac{x}{L}m\right)g}{\frac{m}{L}}} = \sqrt{xg}$$

$$\frac{dx}{dt} = \sqrt{xg} = \sqrt{g} x^{1/2}$$

$$\int_0^L dx x^{-1/2} = \int_0^t \sqrt{g} dt$$

$$\frac{x^{1/2}}{1/2} \Big|_0^L = \sqrt{g} (t - 0)$$

$$\frac{L^{1/2} - 0}{1/2} = \sqrt{g} t$$

$$2\sqrt{L} = \sqrt{g} t$$

$$t = 2\sqrt{\frac{L}{g}}$$

I thought that was kind of a cool result - $2\pi\sqrt{\frac{L}{g}}$ is the period for a mass on the end of a light rope (pendulum), so this is faster than that by a factor of π .